Large-scale machine learning
Stochastic gradient descent

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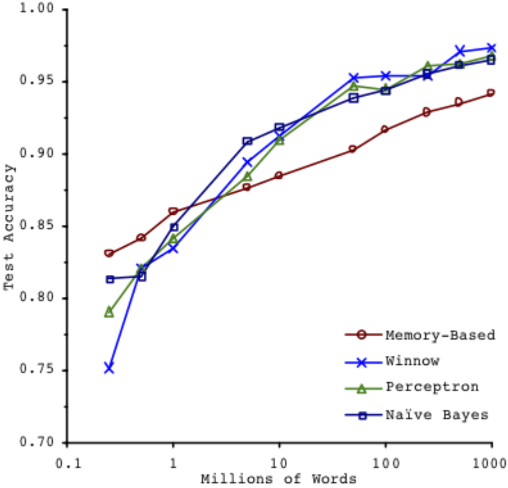
Introduction

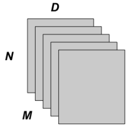
Very large amounts of data being generated quicker than we know what do with it ('08 stats):

- NYSE generates $\sim$ 1 terabyte/day of new trade data
- Facebook has about 1 billion photos $\sim$ 2.5 petabytes
- Large Hadron Collider $\sim$ 15 petabytes/year
- Internet Archive grows $\sim$ 20 terabytes/month

The dynamics of learning change with scale (Banko & Brill '01):



Dealing with large scale: challenges



Could be large:
N *(#data)*, ***D*** *(#features)*, ***M*** *(#models)*

- Data will most likely not fit in RAM
- Disk transfer speed slow compared to its size
  - $\sim$ 3 hrs to read 1 terabyte from disk @ 100MB/s
- 1 Gbit/s ethernet card (125 MB/s) reads about 450 GB/hour

Dealing with large scale: approaches

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- Mix and match!

Review: batch gradient descent

$$J(\theta) = \frac{1}{2}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2.$$

$$\theta_j := \theta_j - \alpha\frac{\partial}{\partial\theta_j}J(\theta).$$

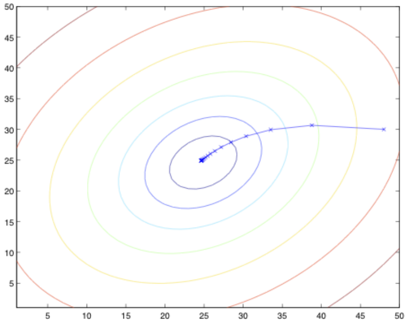
$$\begin{aligned}
\frac{\partial}{\partial\theta_j}J(\theta) &= \frac{\partial}{\partial\theta_j}\frac{1}{2}\left(h_\theta(x) - y\right)^2 \\
&= 2\cdot\frac{1}{2}\left(h_\theta(x) - y\right)\cdot\frac{\partial}{\partial\theta_j}\left(h_\theta(x) - y\right) \\
&= \left(h_\theta(x) - y\right)\cdot\frac{\partial}{\partial\theta_j}\left(\sum_{i=0}^{n}\theta_i x_i - y\right) \\
&= \left(h_\theta(x) - y\right)x_j
\end{aligned}$$

Review: batch gradient descent

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}



Gradient descent

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

}

}

Algorithm: Stochastic gradient descent

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$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)} \qquad \text{(for every } j\text{)}.$$

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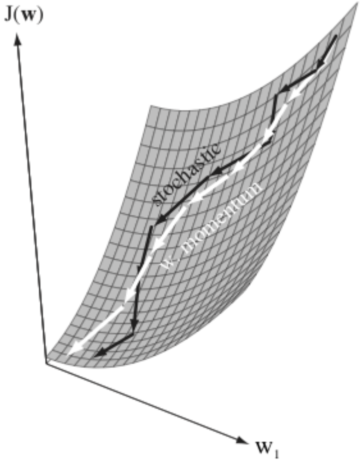
Algorithm: Batch gradient descent

Stochastic gradient descent: what does it look like?



Stochastic gradient descent: why does it work?

$$
\begin{aligned}
\boldsymbol{\mu}_{\mathrm{ML}}^{(N)} &= \frac{1}{N}\sum_{n=1}^{N}\mathbf{x}_n \\
&= \frac{1}{N}\mathbf{x}_N + \frac{1}{N}\sum_{n=1}^{N-1}\mathbf{x}_n \\
&= \frac{1}{N}\mathbf{x}_N + \frac{N-1}{N}\boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)} \\
&= \boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)} + \frac{1}{N}(\mathbf{x}_N - \boldsymbol{\mu}_{\mathrm{ML}}^{(N-1)}).
\end{aligned}
$$

Stochastic gradient descent: why does it work?

- Foundational work in stochastic approximation methods by Robins and Monro, in the 1950's
- These algorithms have proven convergence in the limit, as the number of data points goes to infinity, provided a few things hold:

$$\theta^{(N)} = \theta^{(N-1)} + a_{N-1} z(\theta^{(N-1)})$$

$$\begin{aligned} \lim_{N\to\infty} a_N &= 0 \\ \sum_{N=1}^{\infty} a_N &= \infty \\ \sum_{N=1}^{\infty} a_N^2 &< \infty. \end{aligned}$$

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- In addition, with an unlimited supply of data, stochastic gradient descent is the obvious candidate
- Bottou and Le Cun (2003) show that the best generalization error is asymptotically achieved by the learning algorithm that uses the **most examples within the allowed time**

Small-scale vs large-scale (Bottou & Bousquet '07)

$$
\begin{aligned}
E(\tilde{f}_n) - E(f^*) &= E(f^*_{\mathcal{F}}) - E(f^*) && \text{Approximation error} \\
&+ E(f_n) - E(f^*_{\mathcal{F}}) && \text{Estimation error} \\
&+ E(\tilde{f}_n) - E(f_n) && \text{Optimization error}
\end{aligned}
$$

Problem: Choose $\mathcal{F}$, n, ρ to make error as small as possible, subject to constraints:

- Large scale: constraint is computation time
- Small scale: constraint is number of examples

Small-scale vs large-scale (Bottou & Bousquet '07)

$$\begin{aligned} E(\tilde{f}_n) - E(f^*) &= E(f^*_{\mathcal{F}}) - E(f^*) && \text{Approximation error} \\ &+ E(f_n) - E(f^*_{\mathcal{F}}) && \text{Estimation error} \\ &+ E(\tilde{f}_n) - E(f_n) && \text{Optimization error} \end{aligned}$$

Approximation error bound:
– decreases when $\mathcal{F}$ gets larger.

Estimation error bound:
– decreases when n gets larger.
– increases when $\mathcal{F}$ gets larger.

Optimization error bound:
– increases with ρ.

Computing time T:
– decreases with ρ
– increases with n
– increases with $\mathcal{F}$

Small-scale vs large-scale (Bottou & Bousquet ’07)

- Small scale
  - To reduce estimation error use as many examples as possible
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  - Adjust richness of $\mathcal{F}$

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- Large scale
  - More complicated: computing time depends on all 3 parameters ($\mathcal{F}$, n, ρ)
  - Example: If we choose ρ small, we decrease the optimization error, but we may then also have to decrease $\mathcal{F}$ and/or n, with adverse effects on the estimation and approximation errors

Algorithmic complexity (Bottou '08)

	Cost per iteration	Iterations to reach ρ	Time to reach accuracy ρ
GD	$\mathcal{O}(nd)$	$\mathcal{O}\left(\kappa \log \frac{1}{\rho}\right)$	$\mathcal{O}\left(nd\kappa \log \frac{1}{\rho}\right)$

	Cost per iteration	Iterations to reach ρ	Time to reach accuracy ρ
SGD	$\mathcal{O}(d)$	$\frac{\nu k}{\rho} + o\left(\frac{1}{\rho}\right)$	$\mathcal{O}\left(\frac{d \nu k}{\rho}\right)$

Representative results (Bottou '08)

- **Dataset**
  - Reuters RCV1 document corpus.
  - 781,265 training examples, 23,149 testing examples.
  - 47,152 TF-IDF features.

	Training Time	Primal cost	Test Error
SVMLight	23,642 secs	*0.2275*	*6.02%*
SVMPerf	66 secs	0.2278	6.03%
SGD	1.4 secs	*0.2275*	*6.02%*

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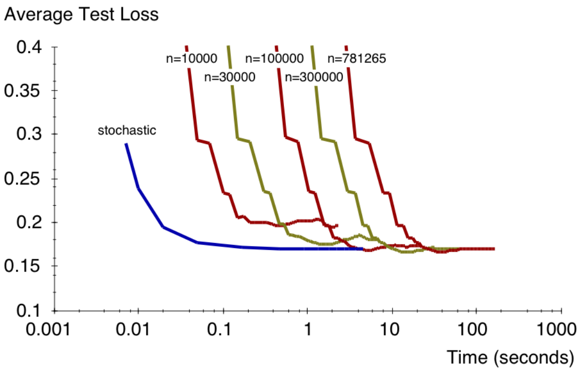

Log-loss problem

Batch Conjugate Gradient on various training set sizes

Stochastic Gradient on the full set

Stochastic gradient descent: usage

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- Code and results for the last 2 @ http://leon.bottou.org/projects/sgd

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- Vowpal Wabbit @ http://hunch.net/∼vw/

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  - Scalable

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- Don't underestimate it!